113 Class Problems: Rings and Fields

1. How many possible ring structures are there on a set with two elements? How about three?

   Solutions:

$R = \{a, b\} \Rightarrow R$ non-trivial $\Rightarrow 1_R \neq 0_R \Rightarrow$ either $a = 0_R, b = 1_R$ or $a = 1_R, b = 0_R$

In both cases the ring axioms force unique $+$ and $\times$, giving $R \cong (\mathbb{Z}/2\mathbb{Z}, +, \times)$. $\Rightarrow$ There are two possible ring structures

$R = \{a, b, c\} \Rightarrow (R, +)$ cyclic of order 3 $\Rightarrow 1_R \neq 0_R$ and 1_R generates R under addition. The ring axioms again force unique $+$ and $\times$, giving $R \cong (\mathbb{Z}/3\mathbb{Z}, +, \times)$. There are 6 possible choices therefore: 3 for 0_R, followed by 2 for 1_R. E.g. $a = 0_R, b = 1_R, c = 1_R + 1_R$

2. Prove that if R is a non-trivial ring then $0_R \notin R^*$.

   Solutions:

$0_R \in R^* \Rightarrow \exists\, a \in R$ such that $a 0_R = 0_R a = 1_R$

$\Rightarrow 0_R = 1_R \Rightarrow R$ trivial.

Hence R non-trivial $\Rightarrow 0_R \notin R^*$

3. If $R_1, R_2, \cdots R_n$ are rings, then the direct product ring is the cartesian product

$$R_1 \times R_2 \times \cdots \times R_n$$

with term by term addition and multiplication. Are the following true:

(a) $(R_1 \times R_2)^* = \{(x_1, x_2) | x_1 \in R_1^*, x_2 \in R_2^*\}$.

(b) R_1 and R_2 fields $\Rightarrow R_1 \times R_2$ a field.

(c) R_1 and R_2 integral domains $\Rightarrow R_1 \times R_2$ an integral domain.

(d) R finite $\Rightarrow (R, +)$ cyclic?

Solutions:

a) True

$(x_1, x_2) \in (R_1 \times R_2)^* \iff \exists (y_1, y_2) \in R \times R$ such that

$(x_1 y_1, x_2 y_2) = (y_1 x_1, y_2 x_2) = (1_{R_1}, 1_{R_2})$

$\iff x_1 \in R_1^*, \ x_2 \in R_2^*$

b) False

Example $(1,0) \in \mathbb{Q} \times \mathbb{Q}$ $(1,0) \neq (0,0)$ but $(1,0) \notin (\mathbb{Q} \times \mathbb{Q})^*$

c) False

Example $(1,0), (0,1) \neq (0,0)$ in $\mathbb{Z} \times \mathbb{Z}$ but $(1,0)(0,1) = (0,0)$

d) False

Example $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$